# A COMPARISON OF CHAPMAN-RICHARDS AND JOHNSON-SCHUMACHER SPLIT-PLOT DESIGN MODELS

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**Abstract.** In this research, intrinsically nonlinear split-plot design models (INSPDMs) are formulated and compared by tailoring the mean part of the split-plot design model (SPDM) to follow Chapman-Richards and Johnson-Schumacher functions. The fitted INSPDMs parameter estimates are obtained by applying the estimated generalized least squares (EGLS) technique. The whole plot and subplot random effects variance components of the INSPDMs are estimated using maximum likelihood estimation (MLE) and restricted maximum likelihood estimation (REML) techniques. The adequacy of the fitted INSPDMs is tested and compared by applying four median adequacy measures (MAM), namely, the resistant coefficient of determination, the resistant prediction coefficient of determination, the resistant modeling efficiency statistic, and the median square error prediction. Also, Akaike's Information Criteria (AIC), Corrected Akaike's Information Criteria (AICC), and Bayesian Information Criteria (BIC) statistics are applied to select the best parameter estimation technique and best model. The results obtained showed that EGLS-REML produced better estimates when compared to EGLS-MLE, and the fitted INSPDM using the Chapman-Richards function is adequate, reliable, stable, and a good fit compared to INSPDM using the Johnson-Schumacher function based on the obtained MAM and IC values.
**Keywords**: *Split-Plot design model, Chapman-Richards function, Johnson-Schumacher function, parameter estimation, information criteria, adequacy measures*

## Introduction

The Split-plot design (SPD) of experiment is a popular experimental design for agronomic trials, industrial experiments, biomedical experiments, and other fields that may require large experimental setup. The design was introduced by Sir R. A. Fishers in 1925 and it has since then metamorphosed into different designs and linear models formation (Gao et al., 2017; Kulahci and Menon, 2017; Anderson, 2016; Anderson and Whitcomb, 2014; Lu and Anderson-Cook, 2014; Lu et al., 2014). Three decades ago, Intrinsically Nonlinear SPD Models (INSPDM) were developed by Blankenship et al. (2003), Knezevic et al. (2002) as well as Gumpertz and Rawlings (1992) recently more studies have emerged as found in the works of John and Michael (2024), David et al. (2023a; 2023b) as well as John et al. (2022a; 2022b). Gumpertz and Pantula (1992) developed and presented the theory of generalized least squares estimation INSPDM parameters as well as the whole plot (WP) and subplot (SP) variance components through the technique of restricted maximum likelihood estimation (REML), maximum likelihood estimation (MLE), analysis of variance (ANOVA), and minimum variance quadratic unbiased estimator (MIVQUE) for unbalanced SPD. In this research, a comparison of two INSPDM, that is, Chapman-Richards and Johnson-Schumacher SPD models are performed using median adequacy measures (MAM) by a researcher and information criteria.

## Materials and Methods

Following the SPD linear model formulation, let (Eq. (1):

$$y_{ijk} = \mu + \gamma_i + \alpha_j + w_{ij} + \beta_k + (\alpha\beta)_{jk} + \varepsilon_{ijk} \qquad \text{Eq. (1)}$$

Be a two factor (A and B) linear SPD model. The corresponding INSPDM general formulation is given as follows (Eq. (2).

$$y_{ijk} = f(x_{ijk}, \theta) + w_{ij} + \varepsilon_{ijk} \qquad \text{Eq. (2)}$$

Where, $y_{ijk}$ is the response variable; *i* = 1, ..., *s* replicates (*Reps*) or block; *j* = 1, ..., *a* levels of the *WP* factor A; *k* = 1, ..., *b* levels of the *SP* factor B; $w_{ij}$ is the *WPE* and $\varepsilon_{ijk}$ is the *SPE*; $f(x_{ijk}, \theta)$ is the nonlinear function for the mean describing the relationship of fixed main and interaction effects to the response $y_{ijk}$. The parameters *Reps*, A and B are assumed fixed. The *WPE* and *SPE* are random effects where they are independent and identically distributed normally as $w_{ij} \sim N(0,\sigma^2_{WP})$ and $\varepsilon_{ijk} \sim N(0,\sigma^2_{SP})$, respectively. If the number of parameters in the mean function, $f(x_{ijk}, \theta)$ is *q* and the number of random effects is *r*, then the number of measurements in the data set, *n*, must be at least *q + r +1* ($n \geq p + r + 1$) in order to estimate all the model's parameters.

### *Split-Plot model with Chapman-Richards and Johnson-Schumacher functions as the mean curve*

The mean curve, $f(x_{ijk}, \theta)$ in equation (2) is substituted with the Chapman-Richards function (CRF) and Johnson-Schumacher function (JSF). Both the CRF and JSF used for this research are three parameter growth functions. Let $f(x_{ijk}, \theta)$ be a CRF and a JSF. Therefore:

$$f(x_{ijk}, \theta)_{CRF} = \alpha_{ijk} \times \left(1 - exp(-\omega x_{ijk})\right)^{\lambda} \qquad \text{Eq. (3)}$$

and

$$f(x_{ijk}, \theta)_{JSF} = \alpha_{ijk} \times exp\left[-\left(\omega\left(\lambda + x_{ijk}\right)^{-1}\right)\right] \qquad \text{Eq. (4)}$$

where, $\alpha_{ijk}$ is the asymptote and it is tailored as $\alpha_{ijk} = \alpha + Rep_i + A_j + B_k + (AB)_{jk}$. Hence, equation (3) and (4) can be rewritten as follows.

$$f(x_{ijk}, \theta)_{CFR} = \left[\alpha + Rep_i + A_j + B_k + (AB)_{jk}\right]\left(1 - exp(-\omega x_{ijk})\right)^{\lambda} \qquad \text{Eq. (5)}$$

and

$$f(x_{ijk}, \theta)_{JSF} = \left[\alpha + Rep_i + A_j + B_k + (AB)_{jk}\right] exp\left[-\left(\omega\left(\lambda + x_{ijk}\right)^{-1}\right)\right] \qquad \text{Eq. (6)}$$

The SPD model with the CRF and JSF as the mean curve is therefore given as follows.

$$y_{ijkl} = \left[\alpha + Rep_i + A_j + B_k + (AB)_{jk}\right]\left(1 - exp\,(-\omega\, x_{ijk})\right)^{\lambda} + w_{ij} + \varepsilon_{ijk} \qquad \text{Eq. (7)}$$

and

$$y_{ijkl} = \left[\alpha + Rep_i + A_j + B_k + (AB)_{jk}\right] exp\left[-\left(\omega\left(\lambda + x_{ijk}\right)^{-1}\right)\right] + w_{ij} + \varepsilon_{ijk} \qquad \text{Eq. (8)}$$

where $\alpha$ is the average yield at zero rate or dose, $Rep_i$ is the $i$th replicate or block, $A_j$ is the effect of the $j$th levels of factor $A$, $B_k$ is the effect of the $k$th levels of factor $B$, $(AB)_{jk}$ is the $j$th and $k$th levels interaction effect of the factors $A$ and $B$ respectively, $x_{ijk}$ is the mean covariate effect in the $i$th replicate at the $j$th factor $A$ and $k$th factor $B$, $\omega$ and $\lambda$ are the Weibull scale and shape parameters respectively, $w_{ij}$ is the WP error and $\varepsilon_{ijk}$ is *SP* error.

### *Method of Estimated Generalized Least Square (EGLS)*

When the covariance matrix of $y$ is known then the GLS estimator, $\hat{\theta}_{\text{GLS}}$ is found by minimizing the objective function (Gumpertz and Rawlings, 1992).

$$\left(y - f(X,\,\theta)\right)^{t} \boldsymbol{C}^{-1}\left(y - f(X,\,\theta)\right) \qquad \text{Eq. (9)}$$

where, $C$ is a known positive definite (non-singular) covariance matrix which arises from Eq. (2). Since $\boldsymbol{C}$ is unknown it needs to be estimated using the methods of REML and MLE to estimate the variance components. The derived estimates are given as;

$$\left\langle tr\left(\hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_i\ \hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_j\right)\right\rangle \times \left\langle\left(\hat{\sigma}^2_{j(h+1)}\right)\right\rangle = \left\langle\left(y^{\mathbf{t}}\hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_i\ \hat{\mathbf{T}}_{(h)}y\right)\right\rangle \qquad \text{Eq. (10)}$$

$$\left\langle\left(\hat{\sigma}^2_{j(h+1)}\right)\right\rangle = \left\langle tr\left(\hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_i\ \hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_j\right)\right\rangle^{-1} \times \left\langle\left(y^{\mathbf{t}}\hat{\mathbf{T}}_{(h)}\hat{\mathbf{C}}_i\ \hat{\mathbf{T}}_{(h)}y\right)\right\rangle \qquad \text{Eq. (11)}$$

and

$$\begin{aligned}\hat{\sigma}^2_{(h+1)} = &\left\langle tr\left(\mathbf{K}_j\mathbf{K}'_j\hat{\mathbf{C}}^{-1}_{(\mathrm{h})}\mathbf{K}_i\mathbf{K}'_i\hat{\mathbf{C}}^{-1}_{(\mathrm{h})}\right)\right\rangle^{-1} \\ &\times\left\langle\left(z_0 - \hat{D}_0(\hat{\theta}^*_{(h+1)} - \hat{\theta}^*_0)\right)' \hat{\mathbf{C}}^{-1}_{(\mathrm{h})}\mathbf{K}_j\mathbf{K}'_j\hat{\mathbf{C}}^{-1}_{(\mathrm{h})}\right. \\ &\left.\times\left(z_0 - \hat{D}_0(\hat{\theta}^*_{(h+1)} - \hat{\theta}^*_0)\right)\right\rangle \end{aligned} \qquad \text{Eq. (12)}$$

The solutions to the equations may turn out to be negative when further iteration does not improve the log-likelihood. In such a case, the negative value is changed to zero before the next iteration.

***Median Adequacy Measure (MAM) statistics and Information Criteria (IC)***

Four proposed Median Adequacy Measure (MAM) statistics for assessing the adequacy of linear SPD models and regression models and three IC are used for this research to assess the adequacy and goodness of fit of the fitted MSPDM. The four MAM statistics used are the resistant coefficient of determination ($r_r^2$) proposed by Kvalseth (1985), resistant prediction coefficient of determination (*Pred*-$r_r^2$), resistant modeling efficiency (*RMEF*) and median square error prediction (*MedSEP*). These statistics are called resistant due to their ability to withstand outliers or extreme values and not increase or decrease unnecessarily when a variable is added or removed from the original model. The ICs used are the Akaike IC (AIC), Corrected AIC (CAIC), and Bayesian IC (BIC). All the MAM and IC formulas used can be found in David et al. (2023a; 2023b) and John et al. (2022a; 2022b).

***Resistant coefficient of determination ( $r_r^2$ )***

The statistic to calculate the WP and SP $r_r^2$ values are as follows.

$$r_{r(wp)}^2 = 1 - \left( \overset{n}{\underset{i=1}{M}}\left(|\varepsilon_i|\right)_{WP} \Big/ \overset{n}{\underset{i=1}{M}}\left(|Y_i - \bar{Y}|\right)_{WP} \right)^2 \quad \text{Eq. (13)}$$

$$r_{r(sp)}^2 = 1 - \left( \overset{n}{\underset{i=1}{M}}\left(|\varepsilon_i|\right)_{SP} \Big/ \overset{n}{\underset{i=1}{M}}\left(|Y_i - \bar{Y}|\right)_{SP} \right)^2 \quad \text{Eq. (14)}$$

Where M is the median of the absolute values from $i$ = 1 to $n$ and $\varepsilon_i$ is the fitted models residuals. The statistics (13 and 14) above uses the median instead of the mean in obtaining a coefficient of determination value that is highly resistant to outliers as proposed by Kvålseth (1985), $0 \le r_r^2 \le 1$. However, for nonlinear models the coefficient of determination value can be negative when the fit is worse, that is, $-1 \le r_r^2 \le 1$.

***Resistant prediction coefficient of determination (Pred-$r_r^2$)***

The statistic to calculate the WP and SP Pred- values are as follows:

$${}_{WP}\,Pred - r_r^2 = 1 - \left( M_{i=1}^{n} \left[ \left(|e_i|\right) \Big/ \left(1 - h_{ii}\right) \right]_{WP}^2 M_{i=1}^{n} \left[ \left| Y_{i(WP)} - \bar{Y}_{(WP)} \right| \right]^{-2} \right) \quad \text{Eq. (15)}$$

$${}_{SP}\,Pred - r_r^2 = 1 - \left( M_{i=1}^{n} \left[ \left(|e_i|\right) \Big/ \left(1 - h_{ii}\right) \right]_{SP}^2 M_{i=1}^{n} \left[ \left| Y_{i(SP)} - \bar{Y}_{(SP)} \right| \right]^{-2} \right) \quad \text{Eq. (16)}$$

Where; M is the median of the squared values from $i$ = 1 to $n$, $e_i$ is the residual, $h_{ii}$ is the hat matrix and -1 ≤ Pred-$r_r^2$ ≤ 1. However, for nonlinear models the prediction coefficient of determination value can be negative when the fit is worse, that is, -1 ≤ Pred-$r_r^2$ ≤ 1.

### *Resistant Modeling Efficiency (RMEF)*

The statistic to calculate the WP and SP RMEF values are as follows:

$$\text{RMEF}_{WP} = 1 - \left( \frac{M_{i=1}^{n}\left[\left|Y_i - f\left(X_i \ldots, X_p\right)_i\right|\right]_{WP}}{M_{i=1}^{n}\left(\left|Y_i - \bar{Y}\right|\right)_{WP}} \right)^2 \quad \text{Eq. (17)}$$

$$\text{RMEF}_{SP} = 1 - \left( \frac{M_{i=1}^{n}\left[\left|Y_i - f\left(X_i \ldots, X_p\right)_i\right|\right]_{SP}}{M_{i=1}^{n}\left(\left|Y_i - \bar{Y}\right|\right)_{SP}} \right)^2 \quad \text{Eq. (18)}$$

Where; M is the median of the absolute values from $i$ = 1 to $n$ and $f(X_i \ldots, X_p)_i$ is the model-predicted values. In a perfect fit RMEF would result in a value equal to one. The upper bound is one and the (theoretical) lower bound is negative infinity (– ∞ < RMEF ≤ 1).

### *Median Square Error Prediction (MedSEP)*

The statistic to calculate the WP and SP MedSEP values are as follows:

$$_{WP}MedSEP = \left(n\right)_{WP}^{-1}\left( M_{i=1}^{n}\left[\left|Y_i - f\left(X_i \ldots, X_p\right)_i\right|\right]_{WP} \right)^2 \quad \text{Eq. (19)}$$

$$_{SP}MedSEP = \left(n\right)_{SP}^{-1}\left( M_{i=1}^{n}\left[\left|Y_i - f\left(X_i \ldots, X_p\right)_i\right|\right]_{SP} \right)^2 \quad \text{Eq. (20)}$$

Where; $M$ is the median of the absolute values from $i$ = 1 to $n$ and $f(X_i \ldots, X_p)_i$ is the model-predicted values. A model with the smallest *MedSEP* value is termed as more adequate.

### *Information Criteria (IC) statistics*

In this research, Akaike's Information Criteria (AIC), Corrected AIC (AICC) and Bayesian Information Criteria (BIC) are used for testing the goodness of fit of the models and to complement the results obtained from MAM. The statistic for each criterion is given as follows.

$$\text{AIC} = 2f\left(\hat{\theta}\right) + 2p \quad \text{Eq. (21)}$$


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AICC = $2f\left(\hat{\theta}\right)+\dfrac{2np}{n-p-1}$ Eq. (22)

BIC = $2f\left(\hat{\theta}\right)+p\log(s)$ Eq. (23)

Where; $f$ () is the negative of the marginal log-likelihood function, $\hat{\theta}$ is the vector of parameter estimates, $p$ is the number of parameters, $n$ is the number of observations and $s$ is the number of subjects.

### *Experimental data and analysis procedure*

In this research, a balanced $3^1$ x $4^2$ replicated mixed Level SP experimental design data is used. The WP has two factors which are irrigation (Ir) and rice varieties (Va). The irrigation was administered three different times, 7 days, 14 days and 21 days on four different rice varieties, NERICA 2, NERICA 3, NERICA 4 and NERICA 14. The SP factor is nitrogen fertilizer (N) and it was administered at four different rates, 30kg N ha$^{-1}$, 60kg N ha$^{-1}$, 90kg N ha$^{-1}$ and 120kg N ha$^{-1}$ on each of the four varieties of rice. The aim of the field trial was to determine irrigation effect on the yield of rice. The research was conducted by Institute of Agricultural Research, Ahmadu Bello University, Zaria, at their experimental field station in Kano State, Nigeria. The procedures for analysis are as follows:

(1) Performend a traditional SP experimental design analysis. This was done to see which of the effects are significant because only the significant effects will be included for the main nonlinear model. Another reason is to avoid unnecessary inclusion of factors in the model and to decrease the number of parameter estimates. To achieve this step using SAS software, the Proc Mixed code is used.

(2) Identifying the significant effects and a reanalysis is executed to obtain the parameter estimates in terms of regression model. The reason is to reduce the size of parameters to be estimated for meaningful nonlinear modeling and interpretation of results. At this stage, the main effects, and their significant interaction effects, the WP and SP C are estimated using the MLE and REML methods as implemented in SAS software through Proc Mixed. A total of 11 parameters are estimated including the asymptote, scale and shape parameters. These parameter estimates are used as initial values for the INSPDM under study.

(3) The asymptote, shape, and scale parameters for each of the nonlinear functions used for remodeling the traditional SPD model where estimated using Proc Nlin code in SAS.

(4) The 11 parameter estimates are used as initial estimates for the nonlinear models formulated in this research. The SAS Proc Nlmixed code is used at this stage of the research to obtain the results for EGLS. While the Proc Nlin code is used for obtaining the OLS results.

(5) The residuals obtained from each fitted NSPD models are used to calculate all four median adequacy measures introduced in the research for assessing the adequacy of each fitted models so as to identify which model is a better adequate model.

## Results and Discussion

*Table 1* and *Table 2* presents the Chapman-Richards SPD (CRSPD) and Johnson-Schumacher SPD (JSSPD) models parameter estimates, standard errors, and P-values from the EGLS via MLE and REML. Based on results from *Table 1* for CRSPD model, the parameter estimates obtained from EGLS-MLE estimation techniques are not similar compared to the estimates from EGLS-REML estimation technique. Also, it can be observed that EGLS-MLE produced negative estimates for $\alpha_4$, $\alpha_5$, and $\alpha_6$ while EGLS-REML produced negative estimates for $\alpha_2$ and $\alpha_5$ only. Also, looking at the standard errors of the estimates (SEE) for both techniques, EGLS-REML produced lower SEE except for parameters $\lambda$ and $\hat{\sigma}^2_\delta$. The p-values from both estimation techniques showed that none of the parameter estimates from EGLS-MLE was significant at 5%. However, for EGLS-REML, only $\alpha_1$, $\alpha_2$, $\alpha_4$, and $\lambda$ are not significant at 5%. The results of JSSPD model presented in *Table 2*, shows that the parameter estimates produced from EGLS-MLE and EGLS-REML are not similar. Also, it can be observed that $\alpha_3$, $\alpha_5$, and $\lambda$ are negative but for EGLS-REML, $\alpha_2$, $\alpha_4$, and $\alpha_5$ are negative estimates. The SEE for both estimation techniques shows that, EGLS-REML produced lower SEE except for parameters $\lambda$, $\hat{\sigma}^2_\delta$, and $\hat{\sigma}^2_\varepsilon$. The p-values from both estimation techniques showed that none of the parameter estimates from EGLS-MLE was significant at 5% except for the ***C***, that is $\hat{\sigma}^2_\delta$ and $\hat{\sigma}^2_\varepsilon$. Also, for EGLS-REML it can be observed that only $\alpha_0$ and $\hat{\sigma}^2_\varepsilon$ are significant at 5%.

***Table 1**. Chapman-Richards Split-Plot design model parameter estimates.*

| Parameter | EGLS-MLE | EGLS-REML | Std. Error a | Std. Error b | P-value a | P-value b |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 26.086 | 27.891 | 29.052 | 2.57 | 0.372 | **<.0001** |
| $\alpha_1$ | 6.176 | 0.389 | 5.019 | 0.637 | 0.222 | 0.544 |
| $\alpha_2$ | 4.038 | -2.155 | 4.899 | 1.367 | 0.412 | 0.119 |
| $\alpha_3$ | 0.256 | 0.196 | 0.263 | 0.069 | 0.333 | **0.005** |
| $\alpha_4$ | -0.108 | 0.0342 | 0.228 | 0.071 | 0.638 | 0.629 |
| $\alpha_5$ | -0.0086 | -0.0122 | 0.0112 | 0.0036 | 0.443 | **0.001** |
| $\alpha_6$ | -0.0101 | 0.0241 | 0.0199 | 0.0085 | 0.615 | **0.006** |
| $\omega$ | 0.0078 | 0.435 | 0.0348 | 0.198 | 0.823 | **0.03** |
| $\lambda$ | 0.317 | 26.631 | 0.278 | 35.732 | 0.256 | 0.458 |
| $\hat{\sigma}^2_\delta$ | 0.252 | 217.60 | 0.387 | 71.031 | 0.516 | **0.003** |
| $\hat{\sigma}^2_\varepsilon$ | 16.43 | 6.2370 | 14.657 | 0.948 | 0.265 | **<.0001** |

*Note: Letters a and b represents EGLS-MLE and EGLS-REML, respectively. Bold values imply significance at 5%.*

***Table 2**. Johnson-Schumacher Split-Plot design model parameter estimates.*

| Parameter | EGLS-MLE | EGLS-REML | Std. Error a | Std. Error b | P-value a | P-value b |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 25.991 | 26.26 | 314.8 | 7.1753 | 0.934 | **0.0003** |
| $\alpha_1$ | 5.583 | 0.591 | 91.85 | 0.5975 | 0.952 | 0.33 |
| $\alpha_2$ | 5.53 | -0.569 | 981.50 | 5.4567 | 0.996 | 0.92 |
| $\alpha_3$ | -0.241 | 0.064 | 49.66 | 0.1362 | 0.996 | 0.642 |
| $\alpha_4$ | 0.0152 | -0.0041 | 46.97 | 0.3235 | 0.999 | 0.99 |
| $\alpha_5$ | -0.286 | -0.0033 | 2.554 | 0.0097 | 0.911 | 0.732 |
| $\alpha_6$ | 2.48 | 0.0134 | 1.93 | 0.00802 | 0.202 | 0.099 |
| $\omega$ | 28.12 | 2.712 | 20.33 | 5.1654 | 0.170 | 0.601 |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\lambda$ | -7.00 | 5.552 | 7.783 | 11.344 | 0.371 | 0.626 |
| $\hat{\sigma}_{\delta}^{2}$ | 0.627 | 3.142 | 0.192 | 23.883 | **0.002** | 0.896 |
| $\hat{\sigma}_{\varepsilon}^{2}$ | 7.12 | 6.2981 | 0.223 | 1.172 | **<.0001** | **<.0001** |

*Note: Letters a and b represents EGLS-MLE and EGLS-REML, respectively. Bold values imply significance at 5%*

Comparing the CRSPD model to the JSSPD model in terms of the parameter estimate techniques, it is obvious that the EGLS-REML outperformed the EGLS-MLE because the EGLS-REML produced the lowest SEE for both models. However, the CRSPD model is a better model because it produced the lowest EGLS-REML values of SEE compared to the SEE values from JSSPD model. Also, CRSPD model produced seven significant parameter estimates based on the EGLS-REML model estimates as against two significant estimates from the JSSPD model for the same EGLS-REML model estimates. *Figure 1* and *Figure 2* presents the SEE radar plots for the CRSPD model estimates using EGLS-MLE and EGLS-REML, and JSSPD model estimates using EGLS-MLE and EGLS-REML. *Table 3* shows that the $r_r^2$, *Pred-* $r_r^2$ , *RMEF*, and *MedSEP* values for the EGLS-MLE and EGLS-REML estimated fitted CRSPD Models are different for the WP and SP sub models. Also, it can be seen that the EGLS-REML have the largest $r_r^2$ values of 99.99% and 79.98%, *Pred-*$r_r^2$ values of 99.9999% and 69.49%, and *RMEF* values of 100% and 80.04% with smallest *MedSEP* values of 1.4E-14 and 0.042797 for the WP and SP sub models, respectively. This implies that the EGLS-REML estimated fitted CRSPD model has a larger proportion of variability explained in the data, better prediction power, more efficient and better error prediction strength for the WP and SP sub models by their respective main and interaction effects. However, the WP sub design model have larger $r_r^2$, *Pred-*$r_r^2$ and *RMEF* values and smaller MedSEP values compared to the SP sub design model for all the OLS, EGLS-MLE and EGLS-REML estimated fitted CRSPD model.

***Table 3a**. Median adequacy measures results.*

| CRSPD | $r_r^2$ | | $Pred-r_r^2$ | | RMEF | | *MedSEP* | |
|---|---|---|---|---|---|---|---|---|
| | WP | SP | WP | SP | WP | SP | WP | SP |
| MLE | 0.9838 | 0.5376 | 0.9816 | 0.2953 | 0.9970 | 0.6109 | 0.0002 | 0.2283 |
| REML | **0.9999** | **0.7998** | **0.9999** | **0.6949** | **1** | **0.8004** | **1.4E-14** | **0.0428** |

***Table 3b**. Median adequacy measures results.*

| JSSPD | $r_r^2$ | | $Pred-r_r^2$ | | RMEF | | *MedSEP* | |
|---|---|---|---|---|---|---|---|---|
| | WP | SP | WP | SP | WP | SP | WP | SP |
| MLE | 0.2422 | -0.0504 | 0.1423 | -0.601 | **0.9959** | **0.7372** | 0.3479 | 1.1781 |
| REML | **0.9984** | **0.7825** | **0.9982** | **0.6686** | **0.9984** | **0.7769** | **1.6E-06** | **0.0505** |

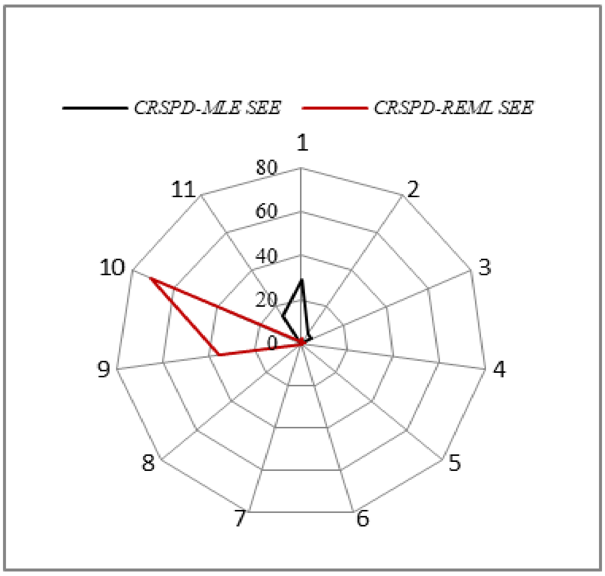


***Figure 1.*** *The SEE Radar Plot for CRSPD Model.*

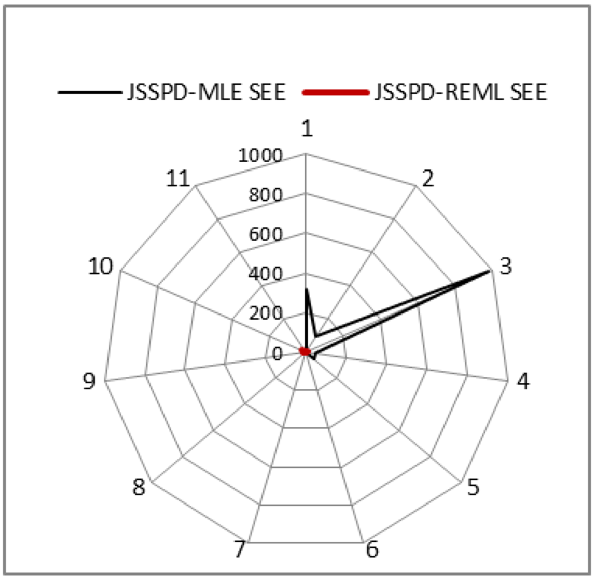


***Figure 2.*** *The SEE Radar Plot for CRSPD Model.*

The estimated (EGLS-MLE and EGLS-REML) fitted JSSPD model adequacy measures for the WP and SP sub design models are also presented in *Table 3*. The results shows that the EGLS-MLE fitted model for both the WP and SP sub design models are worse compared to the EGLS-REML WP and SP sub design models. However, the WP sub design model had larger adequacy measure values compared to the SP sub design model for $r_r^2$ , *Pred-* $r_r^2$ and *RMEF*. While the *MedSEP* values for the WP sub design model is smaller compared to the SP sub design model. Although, all values for the fitted JSSPD models show that a large proportion of variability is explained in the data, high prediction power, better model efficiency and better error prediction strength based on the EGLS-REML fitted models. *Table 4* shows the goodness of fit test (GoFT) outcome based on the Information Criteria (IC) results and it can be seen that for the CRSPD-REML model, the GoFT values produced are smaller than that of the CRSPD-MLE model GoFT values. This implies that the CRSPD-REML model is the best fitted model compared to the CRSPD-MLE model. Also, for the JSSPD-REML model, the GoFT values are smaller than that of the JSSPD-MLE model which implies the JSSPD-REML model is the best fitted model compared to the JSSPD-MLE model. Comparing both models based on their MAM and IC, it can be observed that CRSPD model outperformed JSSPD model because the CRSPD MAM values, that is $r_r^2$, *Pred-* $r_r^2$ and *RMEF* are larger while its *MedSEP* values are smaller for both the WP and SP sub models. Also, the AIC, AICC, and BIC values for CRSPD model are smaller than that of the JSSPD model for both MLE and REML variance component estimated models. These findings are in line with that of the findings from the SEE results obtained and discussed earlier.

***Table 4***. *Information Criteria Test Results.*

| | Method | AIC | AICC | BIC |
|---|---|---|---|---|
| CRSPD | MLE | 553.8 | 557 | 477.6 |
| | REML | **479.6** | **482.8** | **469.7** |
| JSSPD | MLE | 940.2 | 943.3 | 930.2 |
| | REML | **500.4** | **503.6** | **490.5** |

## Conclusion

In this study, two INSPDM are compared for their adequacy, reliability, stability, and goodness of fit when the variance component, ***C*** is estimated through MLE and REML for EGLS model estimation technique. The two INSPDM were developed by tailoring the mean curve of a linear SPD model to follow a Chapman-Richards and Johnson-Schumacher functions. A balanced 2-replicated $3\times4^2$ mixed-level SPD experiment data was analyzed with the two INSPDM and based on the SEE, MAM, and IC values of the two models it was found that the method of EGLS-MLE performed poorly compared to EGLS-REML estimation technique. Also, it was found that the CRSPD model performed better than the JSSPD model. Therefore, the finding suggests that the EGLS-REML produced adequate, stable, efficient, and better parameter estimates for the CRSPD model which is the best fitted model based on the IC values.

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## Conflict of interest

The authors confirm that there is no conflict of interest involve with any parties in this research study.

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